Government of Western Australia
School Curriculum and Standards Authority

MATHEMATICS: SPECIALIST

3C/3D
Calculator-assumed

WACE Examination 2012

Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

When examiners design an examination, they develop provisional marking keys that can be reviewed at a marking key ratification meeting and modified as necessary in the light of candidate responses.

Section Two: Calculator-assumed **(100 Marks)**

Question 8 **(7 marks)**

The equation of a plane Π is $\sqrt{3}x + 2y - 3z = 0$ and the parametric equation of a line L is

$$L = \{x = 0, y = 1 - t, z = -t\}.$$

(a) Find the position of the point of intersection of Π and L. (1 mark)

Solution
$\sqrt{3}(0) + 2(1-t) - 3(-t) = 0$, so $t = -2$. Hence position vector of point of intersection $P = (0, 3, 2)$.
Specific behaviours
✓ solves for the appropriate value of t and so deduces the position of point of intersection

(b) Find the size of the acute angle between L and Π. (3 marks)

Solution
A vector normal to the plane is $\vec{n} = \left(\sqrt{3},\ 2,\ -3\right)$ Line L is parallel to $\vec{u} = \left(0,\ -1,\ -1\right)$ Required angle = $\cos^{-1}\left(\dfrac{\vec{n} \bullet \vec{u}}{\lvert\vec{n}\rvert\lvert\vec{u}\rvert}\right) = \cos^{-1}\left(\dfrac{1}{4\sqrt{2}}\right) = 79.82^\circ \approx 79.8^\circ$ (or 1.39 rad) **OR** Use of CAS: angle$\left(\left[3,\ 2,\ -3\right]\ \left[0,\ -1,\ -1\right]\right) = 79.82° = 79.8°$ (or 1.39 rad) Angle between line and plane = $90° -$ (angle between line and normal) = $10.2°$
Specific behaviours
✓ states a vector normal to plane ✓ calculates the correct angle between line and the normal to the plane ✓ deduces angle between line and plane

(c) Find the distance of the point $Q = (0, 1, -1)$ from the plane Π . (3 marks)

Solution
Let the line S passing through Q and perpendicular to plane Π intersect the plane at M. Parametric equation of line is $(x, y, z) = (\sqrt{3}t, 1+2t, -1-3t)$. Since M lies on the plane, $\sqrt{3}(\sqrt{3}t) + 2(1+2t) - 3(-1-3t) = 0$. Solve using CAS, $t = -\frac{5}{16}$ i.e., solve$\left(\sqrt{3}(\sqrt{3}t) + 2(1+2t) - 3(-1-3t) = 0,\ t\right)$. Shortest distance is $\lvert\overrightarrow{QM}\rvert = \frac{5}{4} = 1.25$ Using CAS, norm$\left(\left[\frac{-5\sqrt{3}}{16}, \frac{-10}{16}, \frac{15}{16}\right]\right) = 1.25$
Specific behaviours
✓ defines that point M on the plane is at shortest distance from Q ✓ correctly solves for t to give where perpendicular line intersects plane Π ✓ determines the shortest distance $\lvert\overrightarrow{QM}\rvert$

Question 9 **(10 marks)**

A hot metal bar is brought to a room of constant temperature. It is known that after the metal has been in the room for t min its temperature y °C is

$$y = 23 + Ae^{-0.04t}$$

for some constant A.

(a) What is the temperature of the room? (1 mark)

Solution
Room temperature = eventual temperature of bar = 23 °C
Specific behaviours
✓ states that the room temperature is 23 °C

(b) Show that y satisfies the differential equation (3 marks)

$$\frac{dy}{dt} + 0.04(y - 23) = 0 \ .$$

Solution
$\frac{dy}{dt} = -0.04Ae^{-0.04t}$ Since $e^{-0.04t} = \frac{y-23}{A}$ (from the given equation) $\frac{dy}{dt} = -0.04A\left(\frac{y-23}{A}\right)$ Hence $\frac{dy}{dt} + 0.04(y-23) = 0$
Specific behaviours
✓ differentiates correctly ✓ rearranges the given equation to express $e^{-0.04t}$ in terms of y and A ✓ substitutes $e^{-0.04t} = \frac{y-23}{A}$ to derive the required differential equation.

(c) If the temperature of the bar 10 minutes after it is brought into the room is 115 °C, show that $A \approx 137$. What was the initial temperature of the bar? (2 marks)

Solution
$115 = 23 + Ae^{-0.04\times 10}$ Hence $A = 92e^{0.4} \approx 137.2 \approx 137$ When $t = 0$, temperature of bar = $23 + 137e^{0.04(0)} = 160$ °C
Specific behaviours
✓ solves the equation for $t = 10$ to evaluate A ✓ evaluates y when $t = 0$

(d) What was the instantaneous rate of change in temperature when the bar was brought to the room? (2 marks)

Solution
$\left.\frac{dy}{dt}\right\|_{t=0} = -0.04 \times 137 = -5.48$ °C / min **OR** Since initial temperature is 160 °C, solving $\frac{dy}{dt} + 0.04(160 - 23) = 0$ $\frac{dy}{dt} = -5.48$ °C/min.
Specific behaviours
✓ accurately evaluates $\frac{dy}{dt}$ at $t = 0$ **OR** ✓ substitutes $y = 160$ in the differential equation and solves for $\frac{dy}{dt}$ PLUS ✓ states the answer using appropriate units

(e) Determine the time that elapsed for the temperature of the bar to drop to two-thirds of its initial value. Give your answer to the nearest second. (2 marks)

Solution
Solving $160 \times \frac{2}{3} = 23 + 137e^{-0.04t}$ $t = 12.33$ min t = 12 min 20 s If the more accurate $t = 137.438$ is used, then t = 12 min, 22 s
Specific behaviours
✓ sets up the equation correctly to solve for t ✓ states the answer correct to nearest second

Question 10 **(6 marks)**

Initially the positions and velocities of a target T and two projectiles A and B are given by:

Target T: initial position $r_T = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ km; velocity $v_T = \begin{pmatrix} 7 \\ 10 \\ -3 \end{pmatrix}$ km/h

Projectile A: initial position $r_A = \begin{pmatrix} 5 \\ 28 \\ -6 \end{pmatrix}$ km; velocity $v_A = \begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix}$ km/h

Projectile B: initial position $r_B = \begin{pmatrix} -5 \\ 20 \\ 1 \end{pmatrix}$ km; velocity $v_B = \begin{pmatrix} -3 \\ -1 \\ 7 \end{pmatrix}$ km/h.

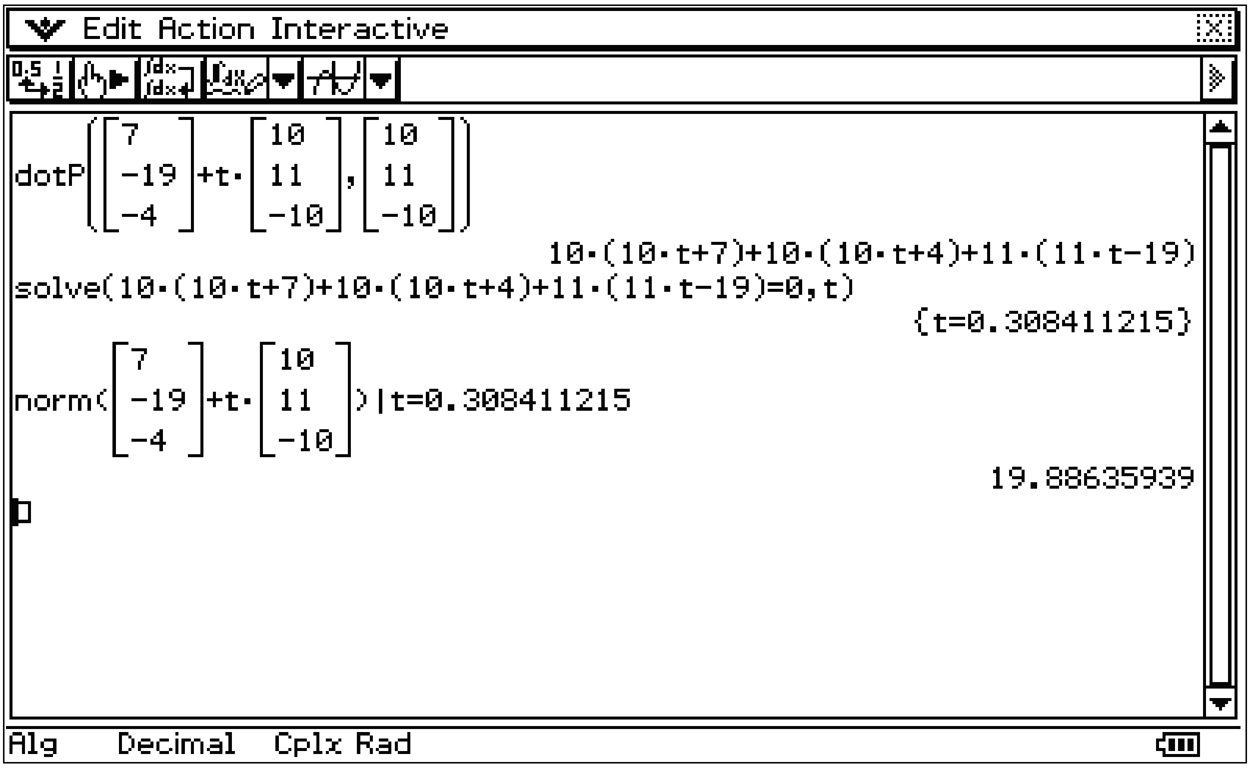
Assuming that all the initial velocities are maintained, determine how many of the projectiles collide with the target. If collisions do occur, find the times and positions of these impacts. If collisions do not occur, state the closest distance, to the nearest kilometre, between the target and the projectile.

Solution
${}_A\vec{V}_T = \begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 7 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ -9 \\ 1 \end{pmatrix}$ and $\overrightarrow{AT} = \begin{pmatrix} -3 \\ -27 \\ 3 \end{pmatrix}$
A collides with T if $t\,{}_A\vec{V}_T = \overrightarrow{AT}$ for some t. Since $t\begin{pmatrix} -1 \\ -9 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -27 \\ 3 \end{pmatrix}$, gives $t = 3$.
$\therefore$ A collides with T in 3 hours.
Position vector for collision = position of T at collision $= \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + 3\begin{pmatrix} 7 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} 23 \\ 31 \\ -12 \end{pmatrix}$ km.
${}_B\vec{V}_T = \begin{pmatrix} -3 \\ -1 \\ 7 \end{pmatrix} - \begin{pmatrix} 7 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} -10 \\ -11 \\ 10 \end{pmatrix}$ and $\overrightarrow{BT} = \begin{pmatrix} 7 \\ -19 \\ -4 \end{pmatrix}$
Since $t\begin{pmatrix} -10 \\ -11 \\ 10 \end{pmatrix} \neq \begin{pmatrix} 7 \\ -19 \\ -4 \end{pmatrix}$ for any t, projectile B does not strike the target.
Hence only A hits the target.
$\vec{d}$ = shortest displacement between Target T and B

$\vec{d} = \overrightarrow{BT} + t\,\overrightarrow{{}_TV_B}$

$\vec{d} \bullet \overrightarrow{{}_TV_B} = 0$

$t = 0.308$ h with shortest distance of 20 km (19.9 km)

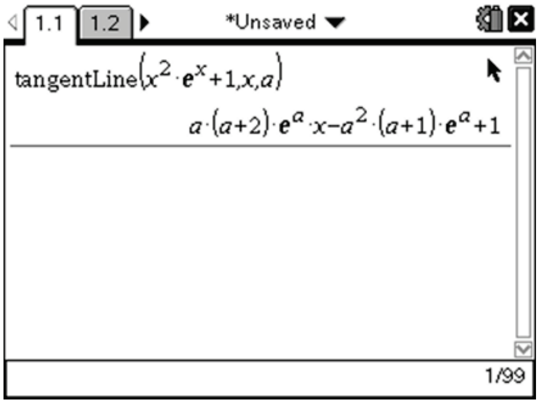


Specific behaviours

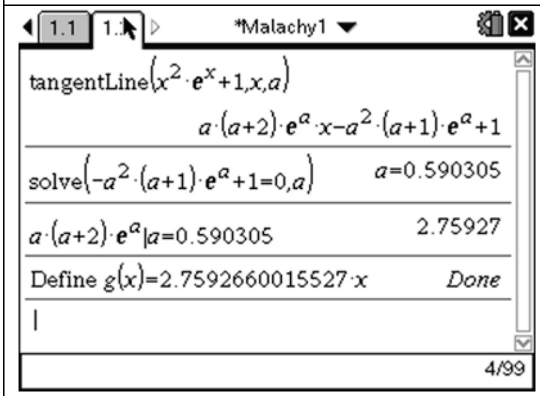
- ✓ correctly determines relative velocity $\overrightarrow{{}_AV_T}$
- ✓ sets the equation $t\,{}_A\vec{V}_T = \overrightarrow{AT}$ for collision condition
- ✓ solves correctly for t
- ✓ states the position vector for collision
- ✓ shows that B does not meet the target at any time
- ✓ states the shortest distance between Target and B

Question 11 **(6 marks)**

(a) Determine the equation of the tangent to the graph of $y = x^2e^x + 1,\ x \geq 0$ at any point where $x = a$. (2 marks)

Solution
$y = a^2e^a + x(a^2 + 2a)e^a - a(a^2 + 2a)e^a + 1$ 
Specific behaviours
✓ determines the correct gradient of the tangent ✓ determines the correct intercept value

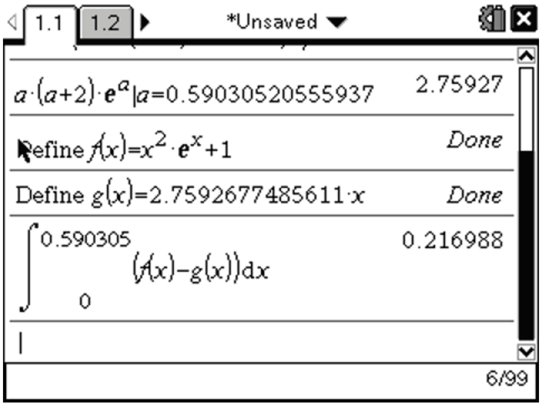
(b) Hence or otherwise, determine the equation of the tangent T which passes through the origin. (2 marks)

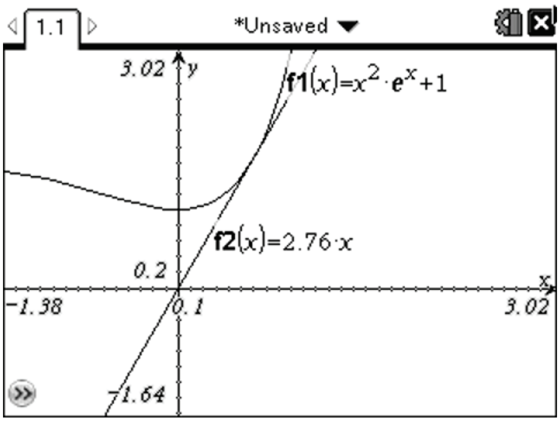
Solution
 $x = \dfrac{((a^3 + a^2)e^a - 1)e^{-a}}{a^2 + 2a} = \dfrac{a^2(1+a) - e^{-a}}{a(a+2)}$
Specific behaviours
✓ solves for a by letting $c = 0$ ✓ correctly defines the gradient of the tangent

(c) Evaluate the area bounded by the y-axis, the graph of $y = x^2e^x+1$ and the tangent T to **four (4)** decimal places. (2 marks)

Solution

$0 = \dfrac{((a^3+a^2)e^a-1)e^{-a}}{a^2+2a}$, $a = 0.5903 = 0.59$ to two decimal places





Specific behaviours

✓ defines the correct integrand
✓ accurately evaluates the defined area

Question 12 **(12 marks)**

The evolution of a small population of female mammals is summarised by the following data:

Age in years	0–1	1–2	2–3	3–4	4–5
Birth rate	0	1.3	1.8	0.9	0.2
Survival rate	0.6	0.8	0.8	0.4	0
Initial population	194	82	55	22	6

In all parts of this question answers should be given to **three (3)** decimal places.

(a) Write down the Leslie matrix L for the population above. (1 mark)

Solution
$\begin{bmatrix} 0 & 1.3 & 1.8 & 0.9 & 0.2 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}$
Specific behaviours
✓ writes the correct Leslie matrix

(b) If t_n denotes the total population after n years, calculate the ratios $\frac{t_{10}}{t_9}$ and $\frac{t_{13}}{t_{12}}$. (3 marks)

Solution
$t_9 = 3653, \quad t_{10} = 4720.6$ $t_{12} = 7882.8, \quad t_{13} = 10186.3$ $\frac{t_{10}}{t_9} = 1.292, \quad \frac{t_{13}}{t_{12}} = 1.292$
Specific behaviours
✓✓ correctly calculates each population ✓ correctly calculates each ratio to three decimal places

(c) Using the results above, what do you notice about the population growth each year? (2 marks)

Solution
$t_{n+1} \approx 1.292t_n$, so population grows at a **steady** rate of 29.2% each year.
Specific behaviours
✓ observes $t_{n+1} \approx 1.292t_n$ ✓ correctly calculates the growth rate

(d) After 13 years, it is decided to cull 10% of all age groups at the beginning of each year. This is equivalent to using a new Leslie matrix $L' = (1 - 0.10)L$. Find the total population at the end of the fifteenth year. (3 marks)

Solution

$$T_{13} = \begin{bmatrix} 5132 \\ 2383 \\ 1475 \\ 913 \\ 283 \end{bmatrix}$$

so $t_{15} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \left(0.9 \begin{bmatrix} 0 & 1.3 & 1.8 & 0.9 & 0.2 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix} \right)^2 \begin{bmatrix} 5132 \\ 2383 \\ 1475 \\ 913 \\ 283 \end{bmatrix} = 13\ 777.5$

∴ The population will be 13 777 at the end of the fifteenth year.

Specific behaviours

✓ calculates T_{13} correctly
✓ uses the correct Leslie matrix to show working
✓ calculates the female population for t_{15}

(e) After 15 years, the population achieved in (d) is considered ideal. The culling rate of 10% will be modified to maintain this population. What new percentage culling rate is needed? (3 marks)

Solution
For stable population, $(1 - h)\ 1.292 = 1$ so $h = 0.226$ hence required culling rate = 22.6%
Specific behaviours
✓ uses the correct growth rate for Part (c) ✓ writes the equation for stable population ✓ calculates the correct culling rate

Question 13 **(9 marks)**

One of the rides at a carnival involves a cabin being oscillated around a centre position with simple harmonic motion. The period of the motion is always 1.5 minutes, but the amplitude can be adjusted.

(a) There is a built-in safety feature that turns off the power when the speed reaches 80 metres/min. Determine the maximum possible amplitude of the motion. (2 marks)

Solution
$v_{\max} = nA$ where A is the amplitude and $n = \frac{2\pi}{1.5}$. $A = \frac{80(1.5)}{2\pi}$ metres
Specific behaviours
✓ correctly finds the angular velocity n ✓ uses the equation to find A

(b) If the amplitude was set at 12 metres, for what percentage of the time would the speed of the cabin be greater than half of the maximum speed stated in Part (a) above? (4 marks)

Solution
Let $x = 12\sin nt$ and $v = 12n\cos nt$ so when $40 = 12\frac{2\pi}{1.5}\cos\frac{2\pi}{1.5}t$ then $t = 0.155\,29$ min. This is the time from the initial position until the velocity first drops below half maximum speed. Hence total time in a period the velocity exceeds the threshold is $0.155\,29 \times 4$ and the percentage of the period is hence $4 \times \frac{0.155\,29}{1.5} = 41.4\ \%$
Specific behaviours
✓✓ correctly sets up the equation to solve for t ✓ uses CAS to solve for t ✓ calculates the required percentage

(c) On a certain day, the amplitude is set to 6 metres but the cabin is observed at $t = 0$ to be at $x = 3$ and moving away from the centre of motion. Derive an expression for the displacement of the cabin from the centre at a time t minutes later. (3 marks)

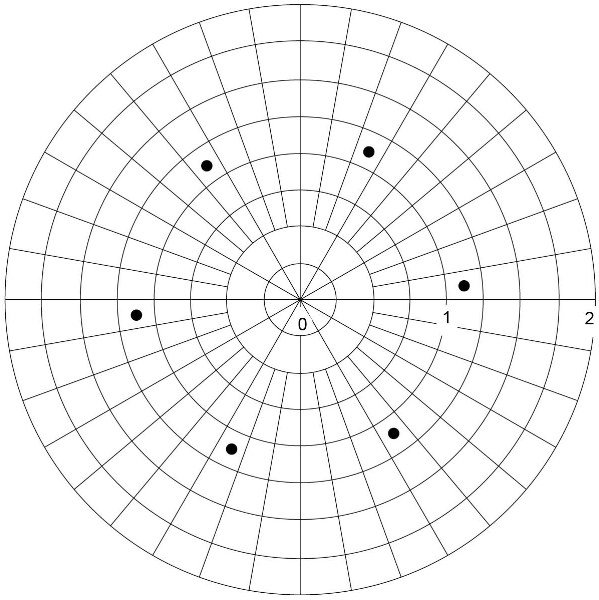
Solution
$x = A\sin\ (nt + \delta)$ $x = 6\sin\left(\frac{2\pi}{1.5}t + \frac{\pi}{6}\right)$ other solutions exist, i.e. cosine etc
Specific behaviours
✓✓ calculates n and δ ✓ writes the correct expression

Question 14 **(8 marks)**

(a) Determine all roots of the equation $z^6 = \sqrt{3} + i$, expressing them in polar form $r\,\text{cis}\,\theta$ where $r \geq 0$ and $-\pi < \theta \leq \pi$. (5 marks)

Solution
$z^6 = 2\text{cis}\,(\frac{\pi}{6} + 2n\pi)$ $z = 2^{\frac{1}{6}}\text{cis}\,(\frac{\pi}{36} \pm \frac{2n\pi}{6})$ $z_1 = 2^{\frac{1}{6}}\text{cis}\,\frac{\pi}{36}$ $z_2 = 2^{\frac{1}{6}}\text{cis}\,\frac{13\pi}{36}$ $z_3 = 2^{\frac{1}{6}}\text{cis}\,\frac{25\pi}{36}$ $z_4 = 2^{\frac{1}{6}}\text{cis}\left(-\frac{35\pi}{36}\right)$ $z_5 = 2^{\frac{1}{6}}\text{cis}\left(-\frac{23\pi}{36}\right)$ $z_6 = 2^{\frac{1}{6}}\text{cis}\left(-\frac{11\pi}{36}\right)$
Specific behaviours
✓ writes z^6 in polar form ✓ uses polar form in working ✓ writes the roots at $\frac{12\pi}{36} = \frac{\pi}{3}$ radians apart ✓✓ writes the remaining four roots within the range specified

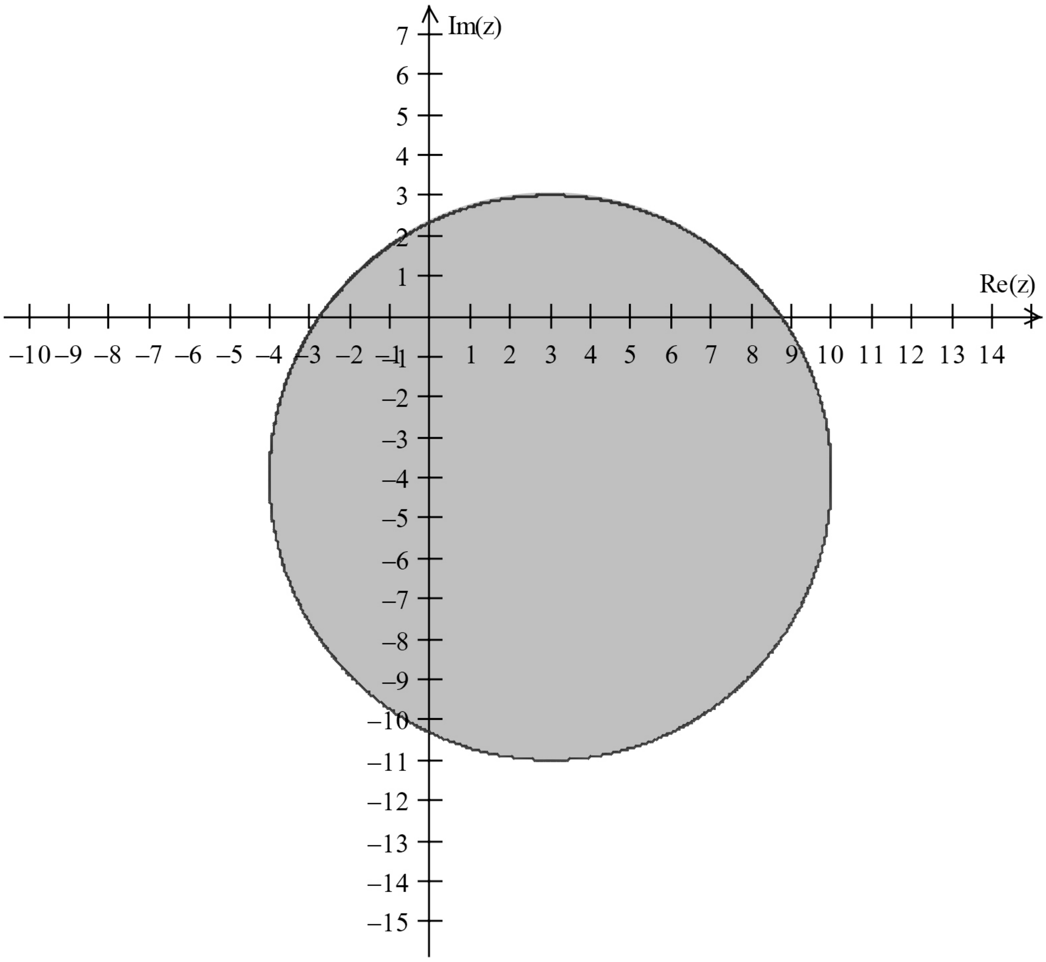
(b) Plot all of the roots above on the diagram provided. (3 marks)

Solution
 Six points on circle of radius $2^{\frac{1}{6}}$ and equally spaced in units of $\frac{\pi}{3}$ radians.
Specific behaviours
✓ shows each root has magnitude $2^{\frac{1}{6}} \approx 1.12$ ✓ shows that first root has argument $\frac{\pi}{36}$ ✓ accurately spaces the other roots at intervals of $\frac{\pi}{3}$ radians.

Question 15 **(10 marks)**

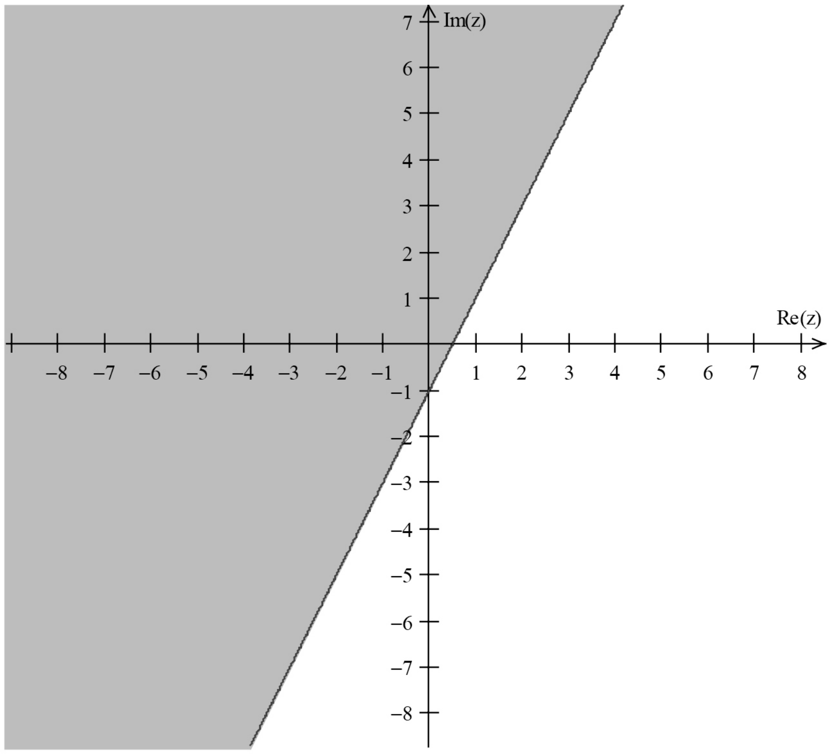
(a) Sketch the following regions in the complex plane.

(i) $|z - 3 + 4i| \leq 7$ (3 marks)

Solution
 shaded circle centre (3, –4) radius 7

Specific behaviours
✓ shape of circumference is close to circular ✓ correctly shades interior of the circular disk ✓ circumference passes through at least three of:(3, 3), (3, –11),(0, –10.3), (8.7, 0) or notes (3, –4) as centre

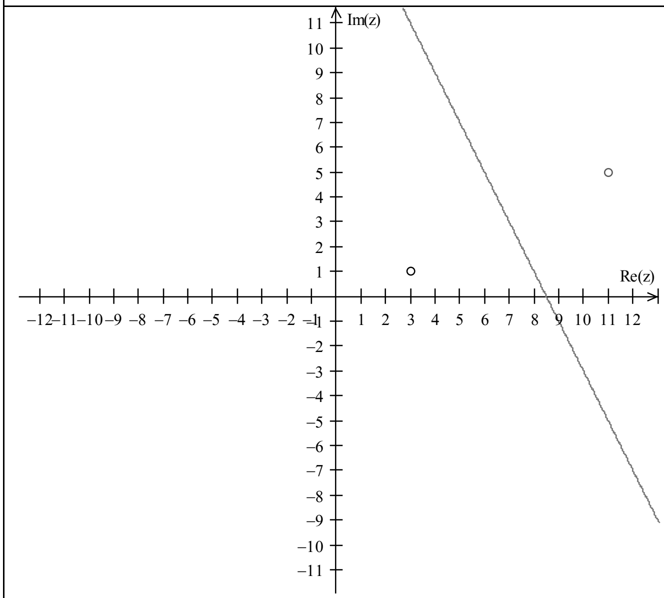
(ii) $\text{Im}\, z \geq 2\, \text{Re}\, z - 1$ (3 marks)

Solution
 shade above line $y = 2x - 1$ in complex plane
Specific behaviours
✓✓ correctly draws the line $y = 2x - 1$ (one mark for the slope and one for an intercept). ✓ shades the region above the line

(b) The set of points in the complex plane that satisfy $|z-3-i| = |z-a-bi|$, where a and b are certain real constants, can alternatively be defined by the property that they lie on the line $\text{Im } z = -2 \text{ Re } z + 17$.

Determine the values of a and b. (4 marks)

Solution

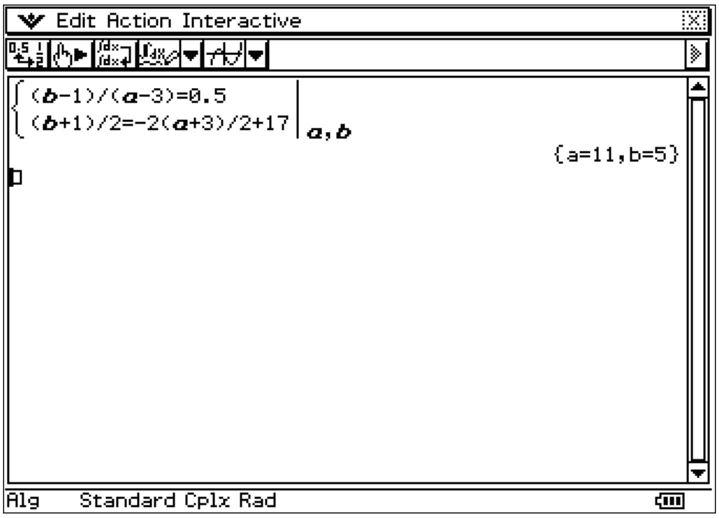


Line joining (3, 1) and (a, b) must have gradient 0.5 (perpendicular to $y = -2x + 17$)

$$\frac{b-1}{a-3} = 0.5$$

Midpoint lies on line $\dfrac{b+1}{2} = -2\dfrac{a+3}{2} + 17$

Solve simultaneous equations for a and b

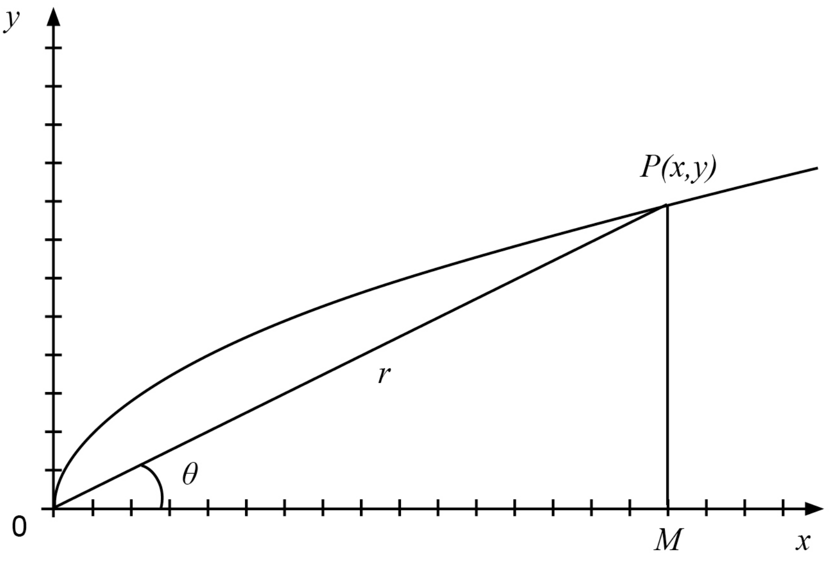


Specific behaviours

- ✓ states line joining (a, b) and (3,1) is perpendicular to $y = -2x + 17$
- ✓ uses the gradient of the line joining (a, b) and (3,1) to obtain equation 1 in a and b
- ✓ uses the midpoint of the line joining (a ,b) and (3,1) to obtain equation 2 in a and b
- ✓ uses CAS correctly to solve the simultaneous equations involving a and b

Question 16 **(8 marks)**

A particle moves along the parabolic path $y^2 = x$ in a way such that the y-coordinate of its position increases at a constant rate of 10 m/s. It starts at the origin O and t seconds later is located at the point $P(x,y)$.



When at P, the particle is at a distance r metres from O and the line OP is inclined at an angle θ radians to the horizontal.

(a) Show that $\tan\theta = \dfrac{1}{y}$ and $r = y\sqrt{1+y^2}$. (3 marks)

Solution
$\tan\theta = \dfrac{y}{y^2} = \dfrac{1}{y}$ $r = \sqrt{y^4 + y^2} = y\sqrt{1+y^2}$
Specific behaviours
✓ correctly expresses $\tan\theta$ in terms of y and x ✓ uses Pythagoras to show $r = \sqrt{x^2 + y^2}$ ✓ replaces x^2 and simplifies expression

(b) When the particle is 2 metres above the horizontal line OM

(i) at what rate is θ changing? (3 marks)

Solution
$\sec^2\theta\dfrac{d\theta}{dt} = -\dfrac{1}{y^2}\dfrac{dy}{dt} = -\dfrac{5}{2}$ $\sec^2\theta = 1+\left(\dfrac{1}{2}\right)^2 = \dfrac{5}{4}$ so $\dfrac{d\theta}{dt} = -2$ rad/s
Specific behaviours
✓ correctly differentiates from $\tan\theta = \dfrac{1}{y}$ implicitly with respect to t ✓ substitutes $\dfrac{dy}{dt} = 10$, and $\sec\theta = \dfrac{\sqrt{5}}{2}$ ✓ correctly solves for $\dfrac{d\theta}{dt}$

(ii) at what rate is its distance from O changing? (2 marks)

Solution
$\dfrac{dr}{dt} = \dfrac{1+2y^2}{\sqrt{1+y^2}}\dfrac{dy}{dt} = \dfrac{9}{\sqrt{5}}\times 10 = 40.25$ m/sec.
Specific behaviours
✓ correctly differentiates the expression $r = y\sqrt{1+y^2}$ ✓ correctly evaluates $\dfrac{dr}{dt}$

Question 17 **(5 marks)**

Question 17 **(5 marks)**

Let $y = a^x$ where a is a positive constant.

(a) Determine $\frac{dy}{dx}$. (2 marks)

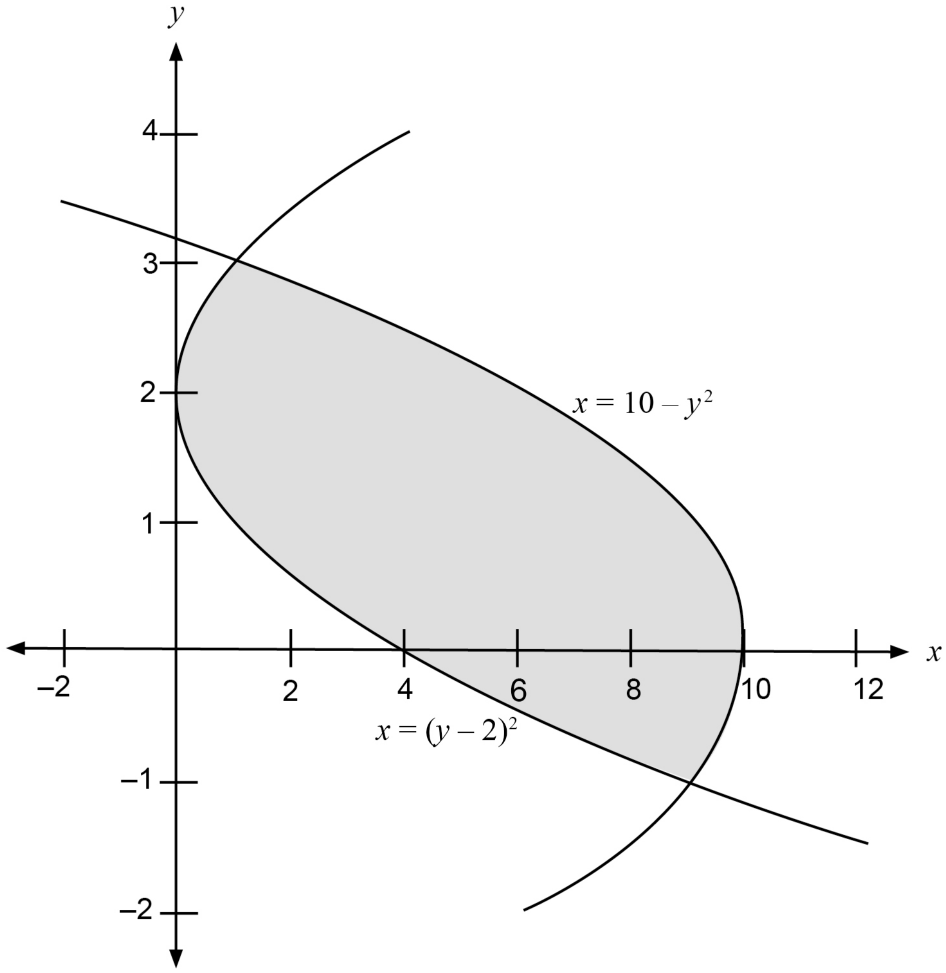
Solution
$y = a^x$ $\ln y = x \ln a$ Differentiating both sides, $\frac{1}{y}.\frac{dy}{dx} = \ln a$ so $\frac{dy}{dx} = y \ln a$
Specific behaviours
✓ takes logarithms of the initial expression ✓ correctly implements logarithmic differentiation

(b) Prove that $\int_0^1 y dx = \frac{a-1}{\ln a}$. (3 marks)

Solution
$\int_0^1 y\ dx = \frac{1}{\ln a}\int_0^1 (\ln a)\, y\ dx$ $= \frac{1}{\ln a}\int_0^1 \frac{dy}{dx}\ dx$ $= \frac{1}{\ln a}\, y\Big]_{a^0}^{a^1} = \frac{a-1}{\ln a}$
Specific behaviours
✓ adjusts the integrand by multiplying constant $\ln a$ to get $\frac{dy}{dx}$ ✓ finds the anti-derivative ✓ evaluates the definite integral

Question 18 **(8 marks)**

(a) Determine the shaded area between the curves below. (4 marks)



Solution
Where curves meet $10 - y^2 = (y-2)^2 \Rightarrow 2y^2 - 4y - 6 = 0$ Then $y^2 - 2y - 3 = (y-3)(y+1) = 0 \Rightarrow y = -1, 3$ Required area is then: $= \int_{-1}^{3} \left\{(10 - y^2) - (y-2)^2\right\} dy$ $= \left[10y - \frac{1}{3}y^3 - \frac{1}{3}(y-2)^3\right]_{y=-1}^{y=3}$ $= 30 - 9 - \frac{1}{3} + 10 - \frac{1}{3} - 9 = 21\frac{1}{3}$ square units
Specific behaviours
✓ solves where the two curves intersect ✓ writes down the correct integral for the required area ✓ integrates the terms correctly ✓ evaluates to find the area

(b) Determine the values of the real constants p,q and r if the equation $|3x+6| = p|x+q| + r$ is satisfied for all $x \in [-2, 3]$ but for no other real values. (4 marks)

Solution
As the equation ceases to hold beyond $x=3$ the absolute value function on the RHS must change sign at this point. Hence $q=-3$. Then, for $-2 \le x \le 3$ the equation becomes $\left\|3x+6\right\| = p\left\|x-3\right\| + r \Rightarrow 3x+6 = p(3-x)+r$ Thus $3x+6 = (3p+r) - px$ Whereupon we deduce that $p=-3,\ r=15$.

Specific behaviours
✓ draw clean sketch of the graphs ✓ uses the gradient to evaluate P ✓ uses the x value of upper limit to evaluate q. ✓ uses the y value of upper limit to evaluate r

Question 19 **(11 marks)**

(a) Prove that $\text{cis}\theta \, \text{cis}\varphi = \text{cis}(\theta + \varphi)$. (You should not assume that $\text{cis}\theta = e^{i\theta}$.) (3 marks)

Solution
$\text{LHS} = (\cos\theta + \text{i}\sin\theta)(\cos\phi + \text{i}\sin\phi)$ $= \cos\theta\cos\phi - \sin\theta\sin\phi + \text{i}(\sin\theta\cos\phi + \cos\theta\sin\phi)$ $= \cos(\theta+\phi) + \text{i}\sin(\theta+\phi)$ $= \text{cis}(\theta+\phi)$ $= \text{RHS}$
Specific behaviours
✓ expands $\text{cis}\theta$ and $\text{cis}\phi$ ✓ groups product into real and imaginary parts ✓ shows that product equates $\text{cis}(\theta+\phi)$

(b) By considering $(5 + i)^4 (1 - i)$, or otherwise, prove that

$$4\tan^{-1}\frac{1}{5} = \frac{\pi}{4} + \tan^{-1}\frac{1}{239}.$$ (4 marks)

Solution
$(5+i)^2 = 24 + 10i$ $(5+i)^4 = (24+10i)(24+10i) = 476 + 480i$ $(5+i)^4(1-i) = (476+480i)(1-i) = 956 + 4i$ Hence $\arg(5+i)^4 + \arg(1-i) = 4\arg(5+i) + \arg(1-i) = \arg(956+4i)$ Thus $4\tan^{-1}\frac{1}{5} - \tan^{-1}1 = \tan^{-1}\frac{4}{956}$ or $4\tan^{-1}\frac{1}{5} = \frac{\pi}{4} + \tan^{-1}\frac{1}{239}$.
Specific behaviours
✓ evaluates $(5+i)^4(1-i)$ in complex form ✓ applies addition of arguments formula ✓ states the appropriate form of the argument of $(5+i)^4$ ✓ deduces the required result

(c) Let $z_1 = \text{cis}\,\theta$ and $z_2 = \text{cis}\,\varphi$, such that $|z_1 - z_2| = |z_1 + z_2|$. (4 marks)

(i) Show that $Re(z_1\overline{z_2}) = 0$.

(ii) Hence, or otherwise, deduce that $|\theta - \varphi| = \frac{\pi}{2}$.

Solution
Given that $\lvert z_1 - z_2\rvert^2 = \lvert z_1 + z_2\rvert^2 \Rightarrow \lvert z_1\rvert^2 - z_1\overline{z}_2 - \overline{z}_1 z_2 + \lvert z_2\rvert^2 = \lvert z_1\rvert^2 + z_1\overline{z}_2 + \overline{z}_1 z_2 + \lvert z_2\rvert^2$ so $2\left(\overline{z}_1 z_2 + z_1\overline{z}_2\right) = 0$ Since $\overline{\left(z_1\overline{z}_2\right)} = \overline{z}_1 z_2$, the sum of $z_1\overline{z}_2$ plus its conjugate is zero so the real part must vanish. Since $\arg(z_1\overline{z}_2) = \theta - \varphi$, the real part of $z_1\overline{z}_2 = 0 \Rightarrow \cos(\theta - \varphi) = 0$. Hence $\theta - \varphi = \pm\frac{\pi}{2}$ or $\lvert\theta - \varphi\rvert = \frac{\pi}{2}$.
Specific behaviours
✓ expands correctly ✓ identifies that $\overline{\left(z_1\overline{z}_2\right)} = \overline{z}_1 z_2$ ✓ concludes that the real part of $z_1\overline{z}_2$ vanishes. ✓ understands if $\text{Re}\left(z_1\overline{z}_2\right) = 0$, then argument is $\frac{\pi}{2}$ and calculates $\arg(z_1\overline{z}_2) = \theta - \varphi$

Published by the School Curriculum and Standards Authority of Western Australia
27 Walters Drive
OSBORNE PARK WA 6017